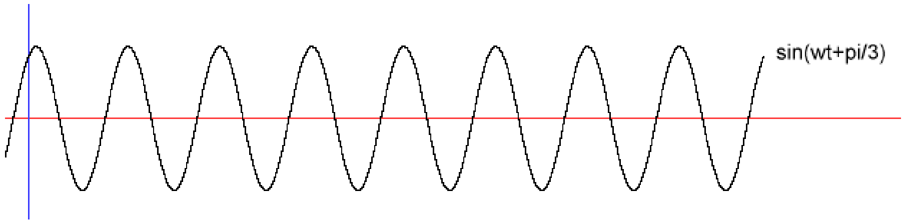
Image Enhancement:

Frequency representation

Dr. Tushar Sandhan

Introduction

- Signal decomposition



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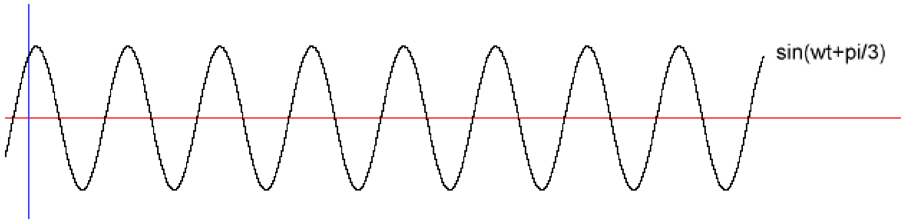

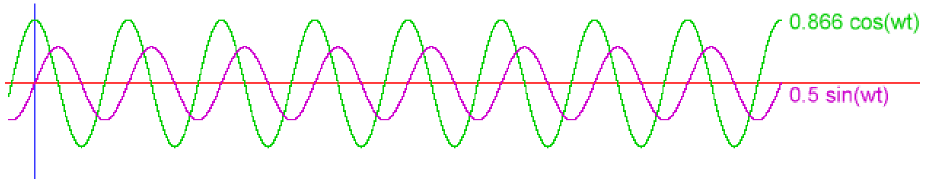

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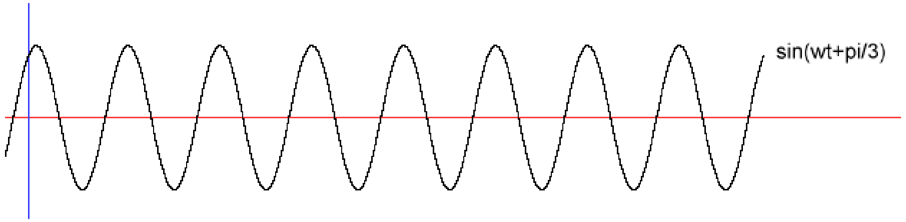

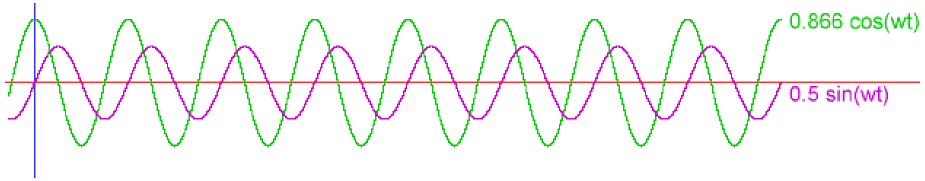

$$\sin(wt + \phi) = \sin(wt)\cos(\phi) + \cos(wt)\sin(\phi)$$

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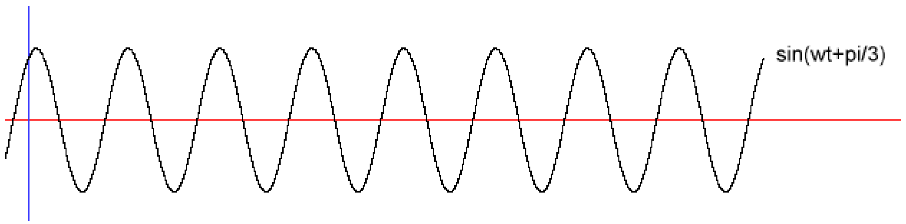

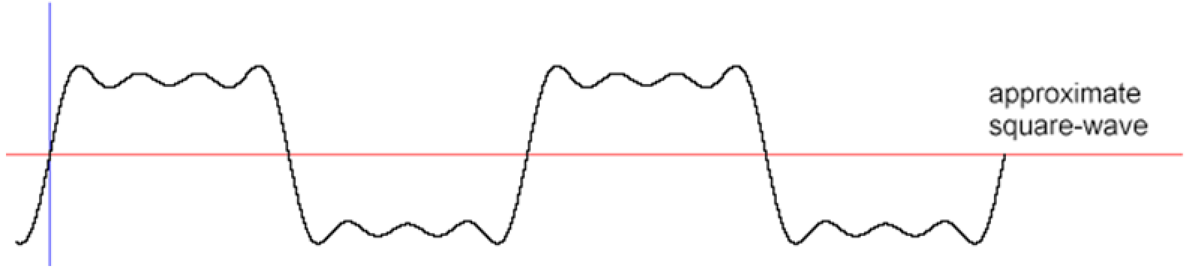

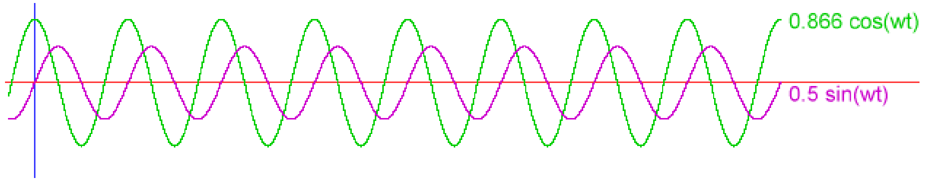

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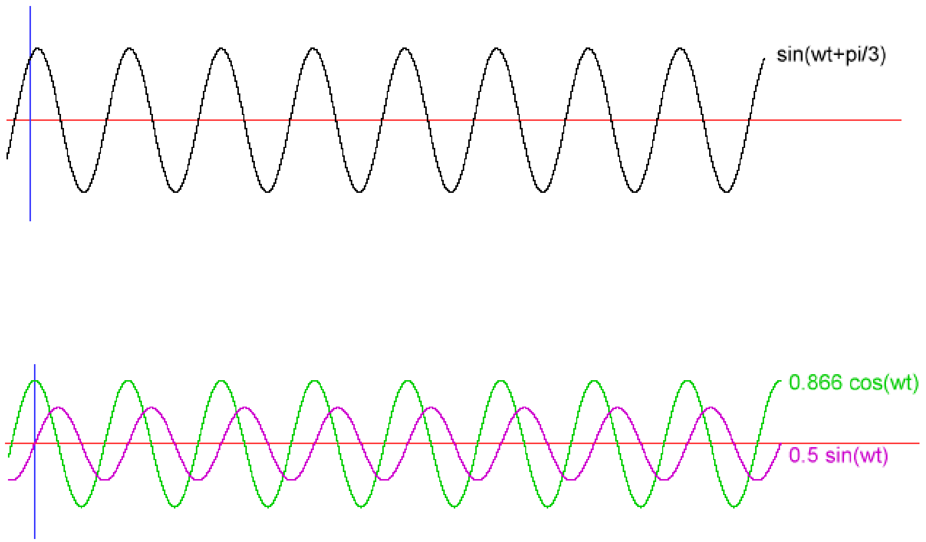

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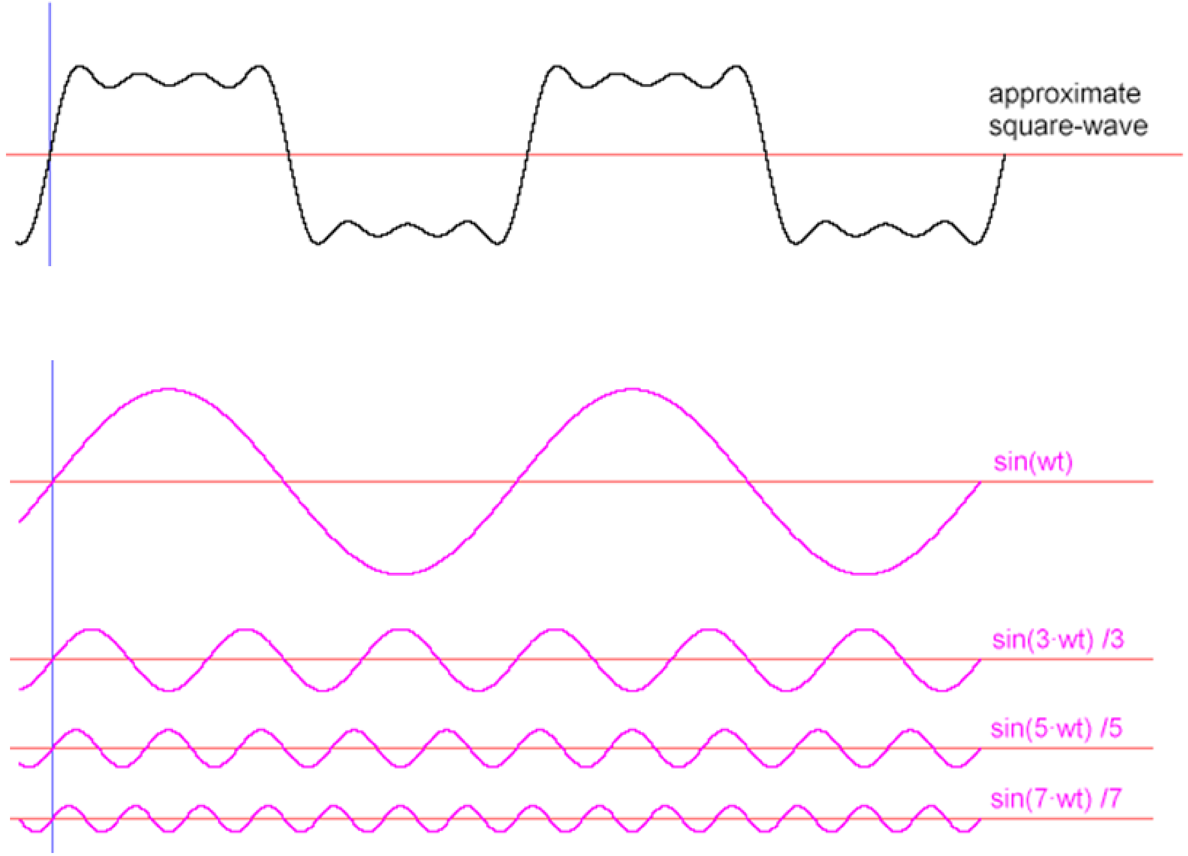

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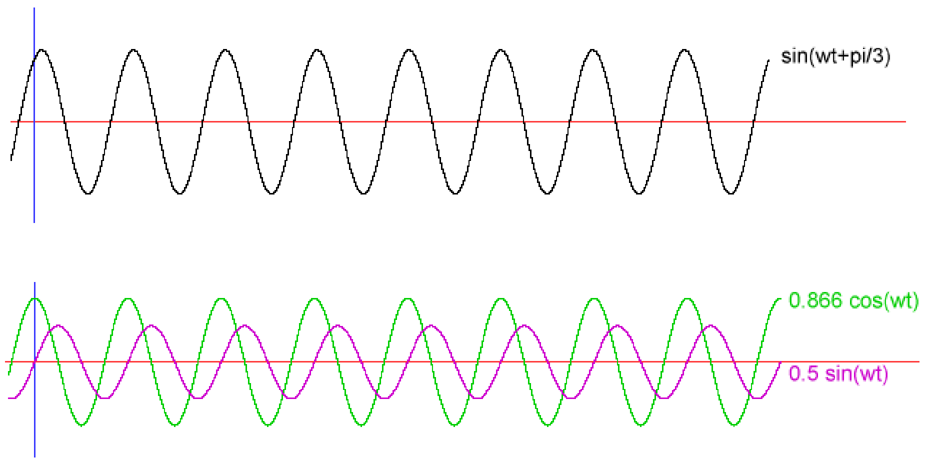

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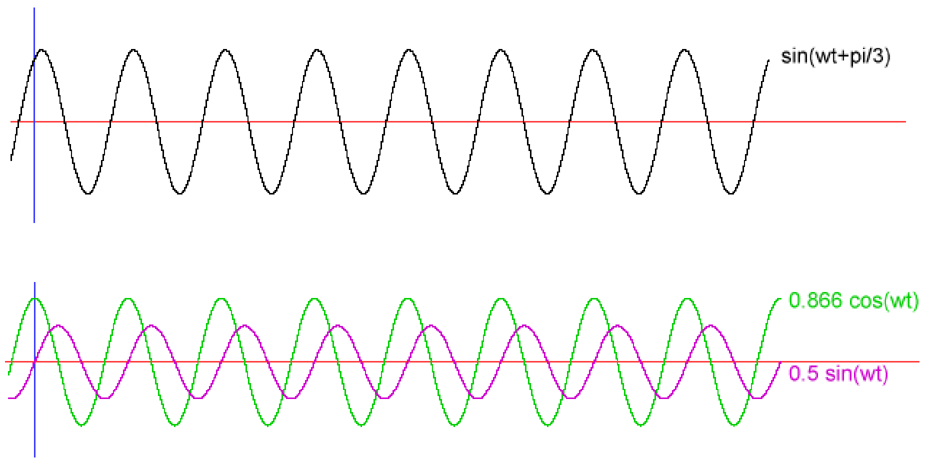

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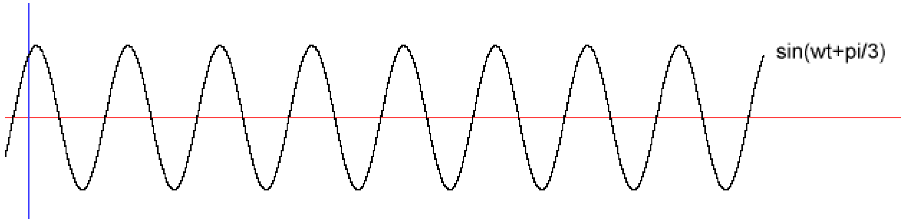

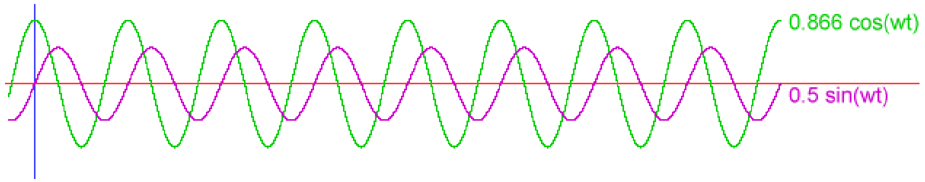

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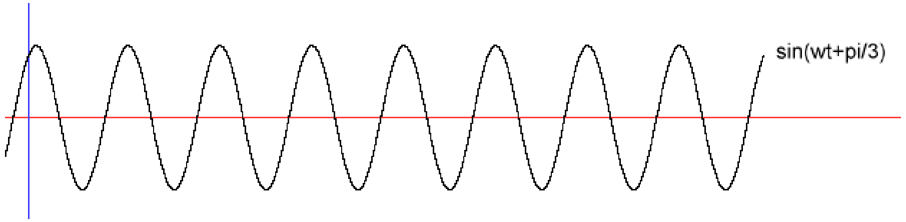

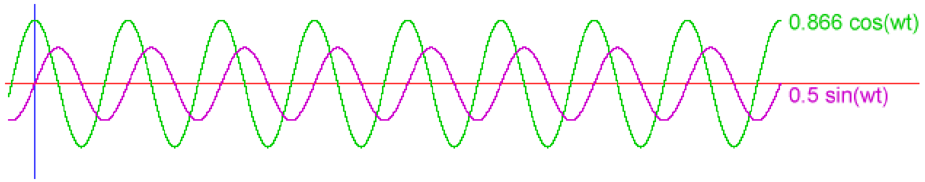

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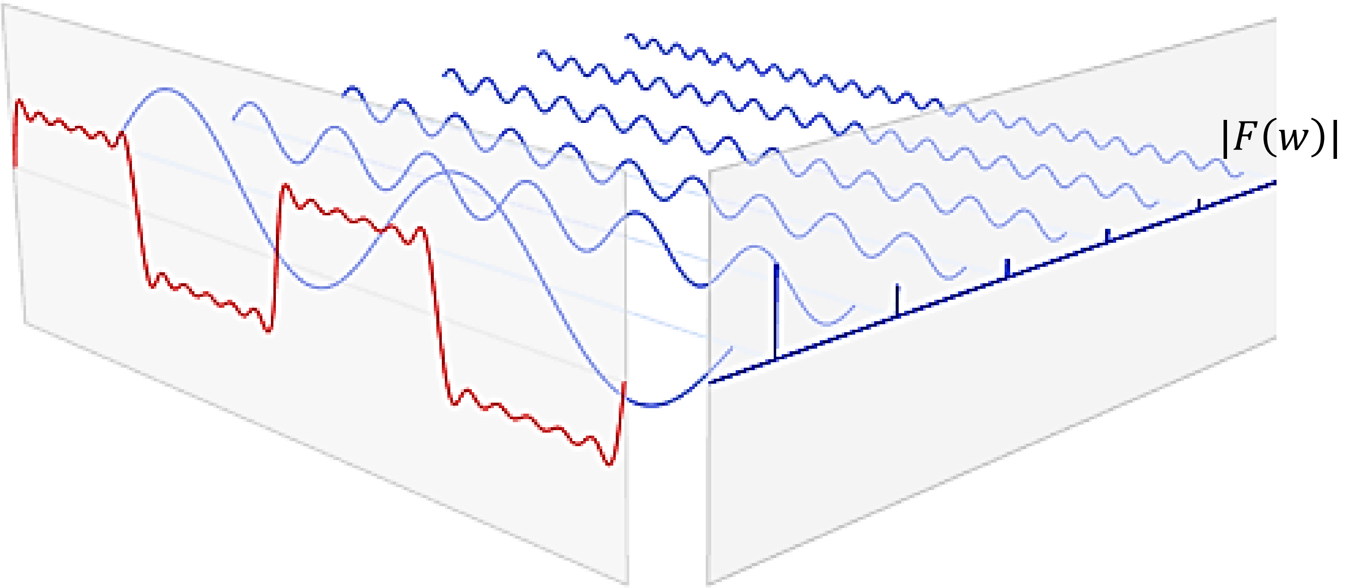

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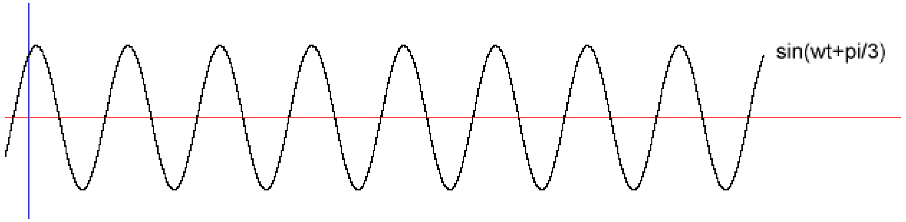

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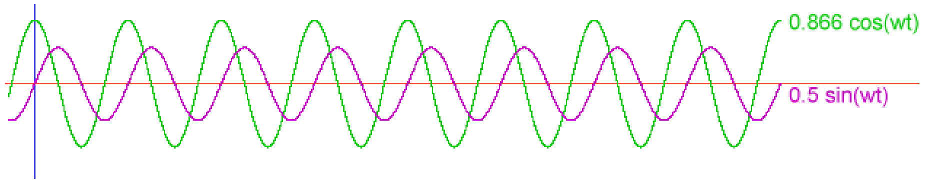

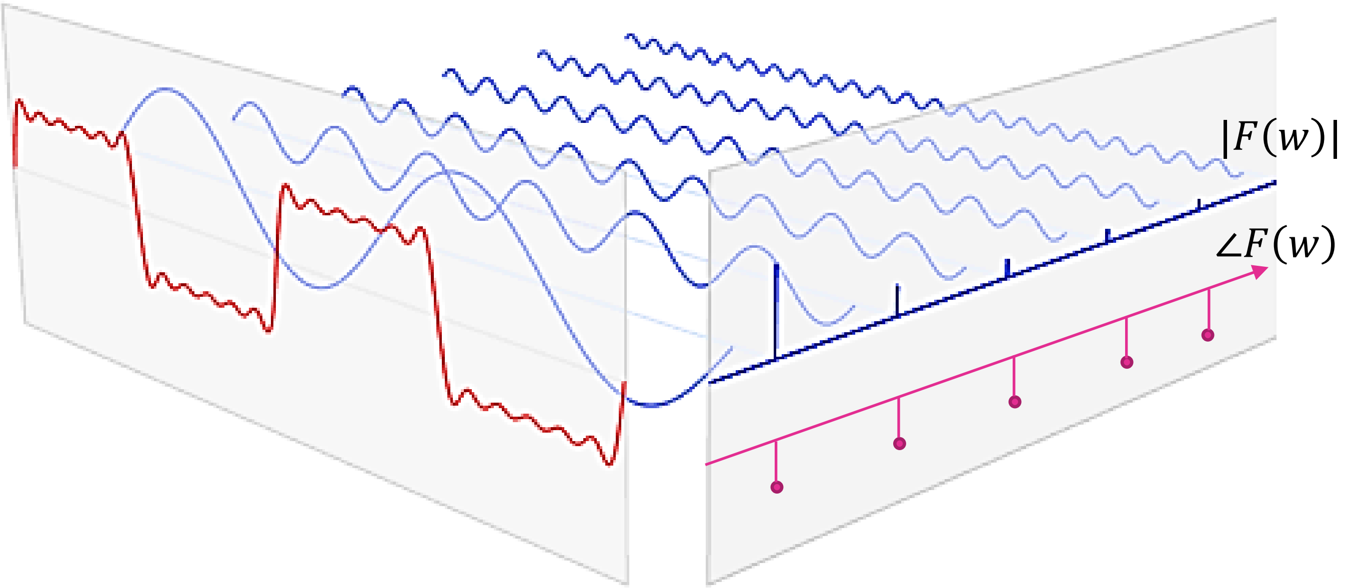

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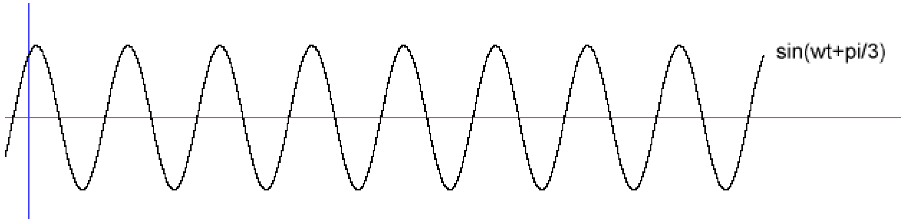

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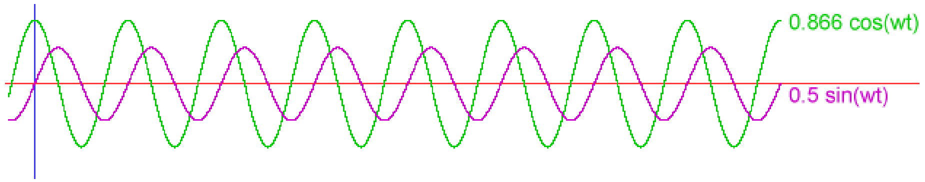

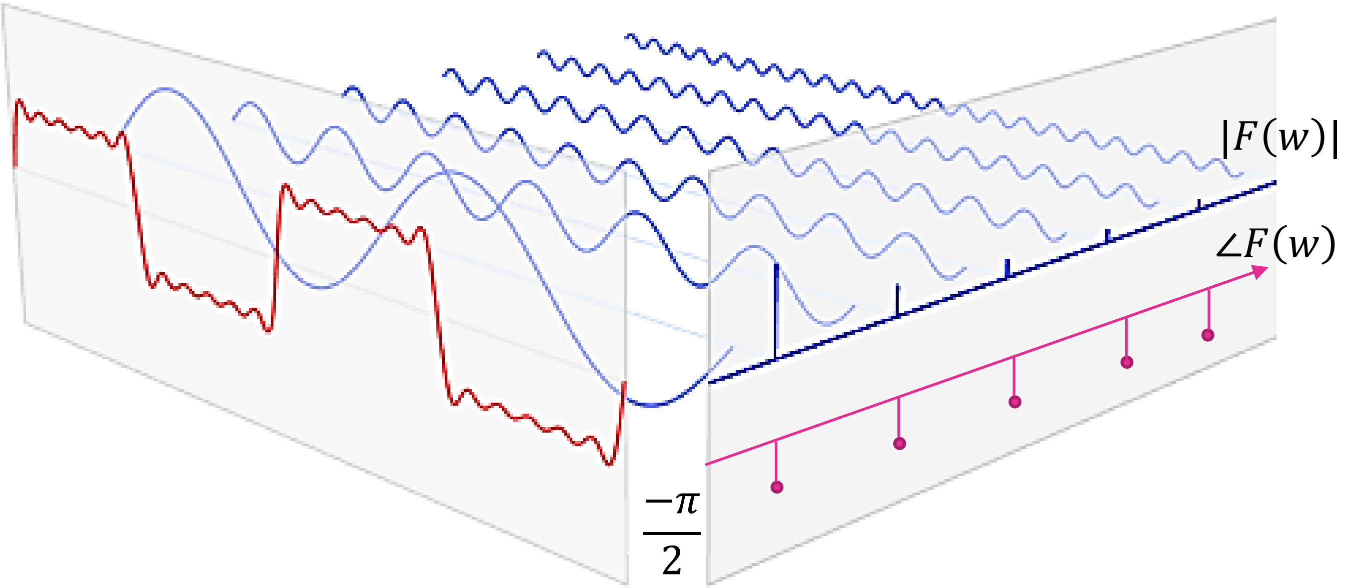

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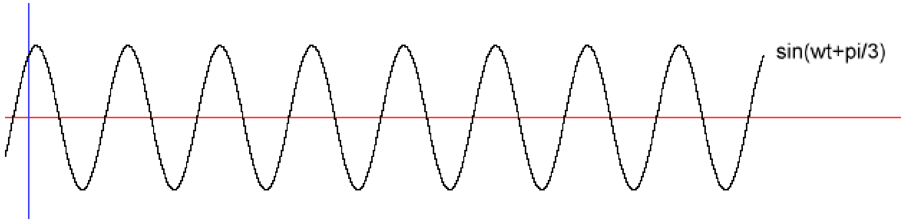

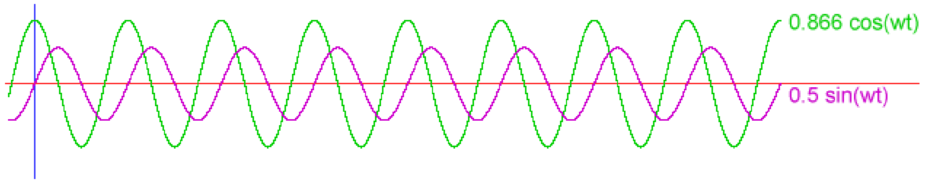

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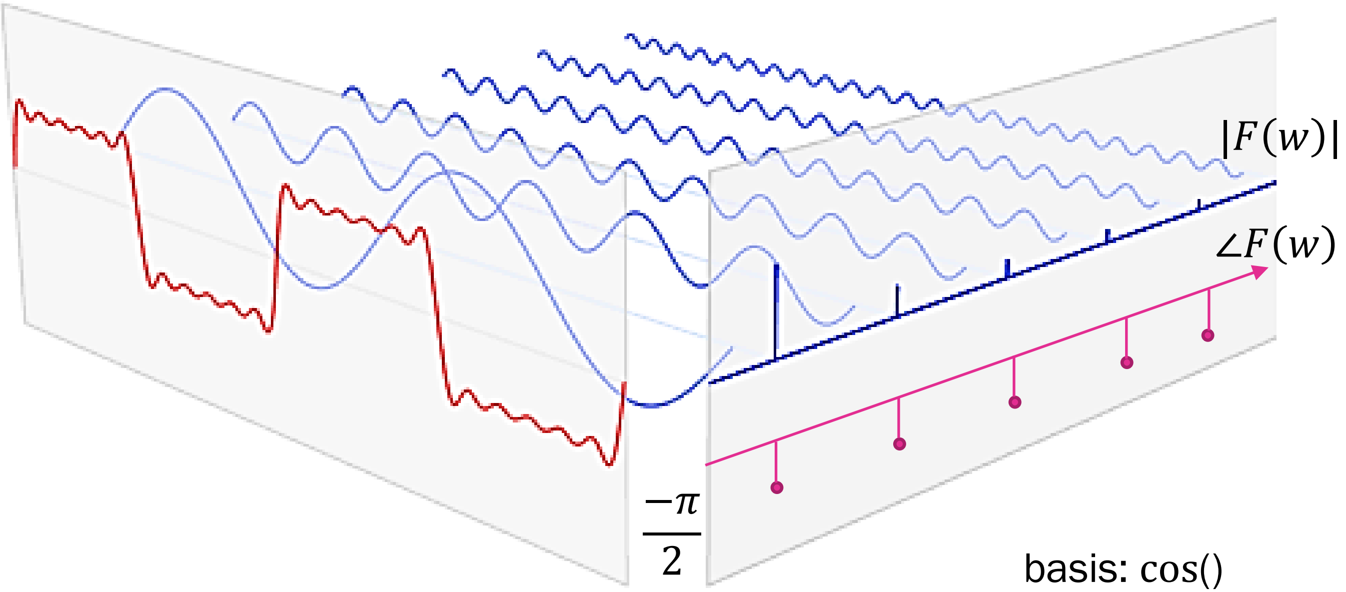
basis: sin()



basis: cos()

Frequency

- 2D harmonics
  - $v_o = 0?$

$$f(x, y) = A\cos(u_0 x + v_0 y + \varphi)$$

Frequency

- 2D harmonics
  - $v_o = 0$?
  - identical sinusoids with
    - amplitude A
    - spatial period $P_x = \frac{2\pi}{u_0}$
    - vertical stripes in the image

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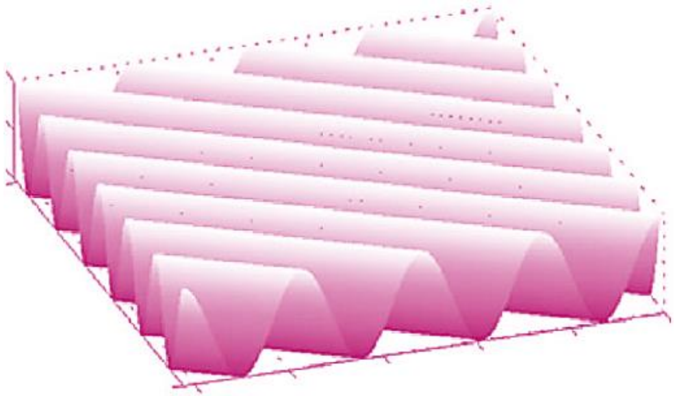

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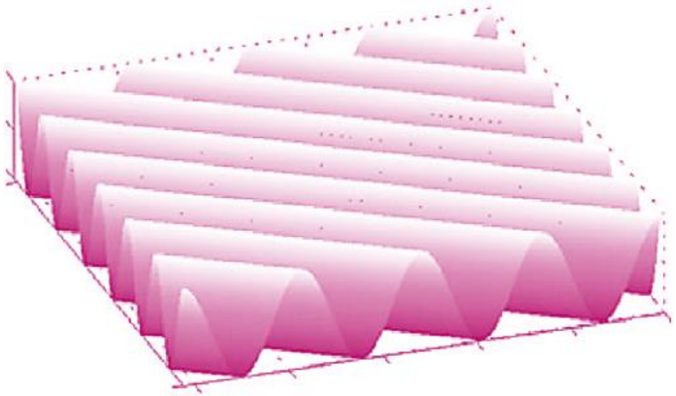

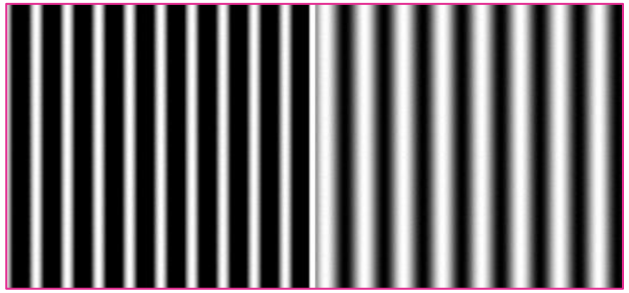

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    - vertical stripes in the image
  - space frequency
    - in both x, y directions
    - x direction: $\frac{u_0}{2\pi}(m^{-1})$
    - number of stripes per meter in x direction

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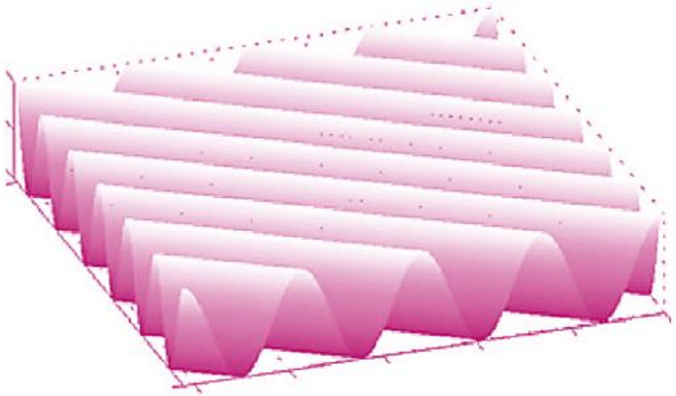

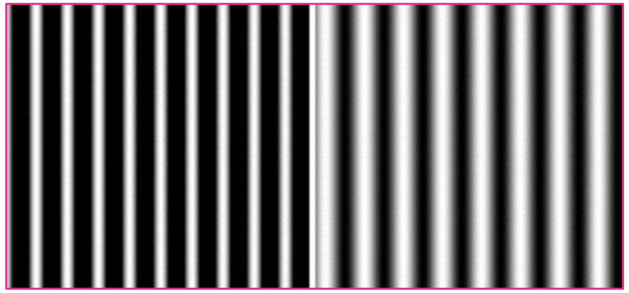

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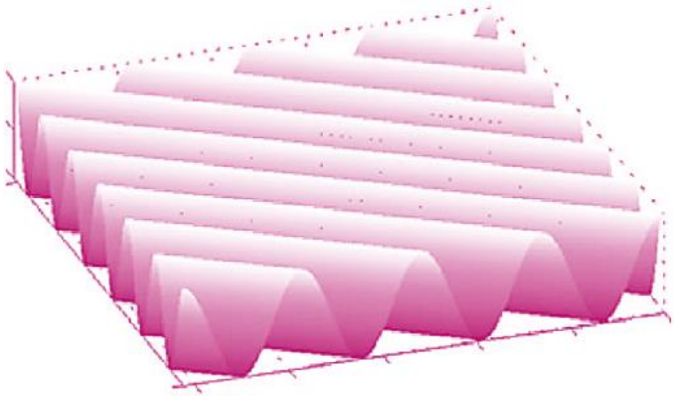
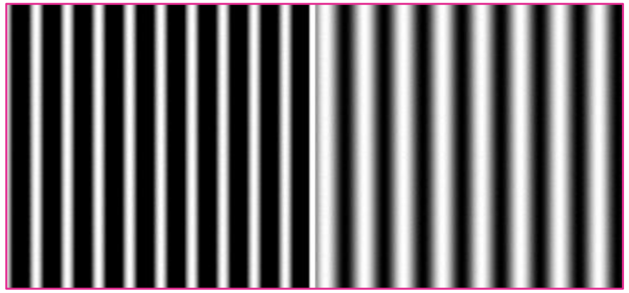


- phase
  - determines shift d_x of ridges from origin of coordinates
  - d_x is from origin of coordinates as a fraction of harmonic's period
  - $d_x = P_x \frac{\phi}{2\pi} = \frac{\phi}{u_0}$



Frequency

- 2D harmonics
  - $u_0 \neq 0$ & $v_o \neq 0$?

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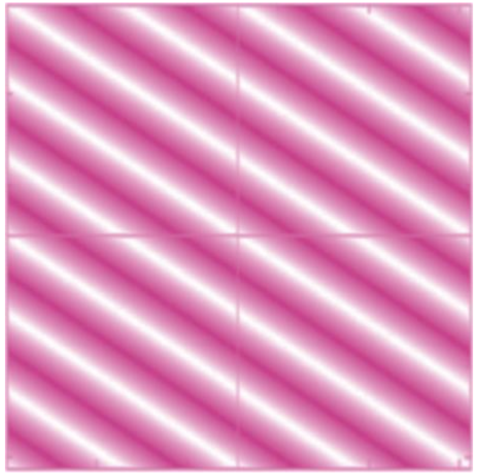

Frequency

- 2D harmonics
  - $u_0 \neq 0 \;\&\; v_o \neq 0$?
  - stripes become oblique
    - They make angel ϑ with x axis
    - $\vartheta = \arctan(\frac{v_0}{u_0})$
    - ratio of both frequencies determines orientation
    - ridges of stripes are characterized by $u_0 x + v_0 y + \varphi = \pm k 2\pi$

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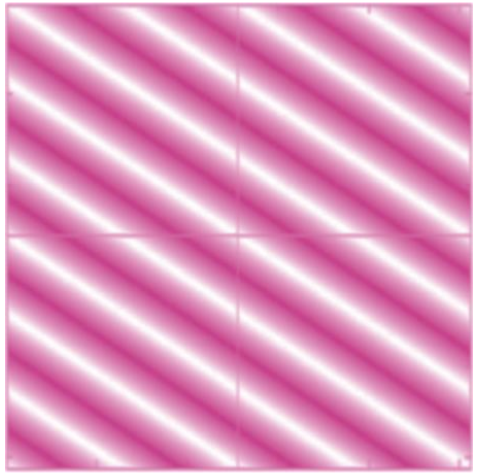

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    - family of linear equations
    - oblique parallel lines
    - distance between them = period P of harmonics
    - $P = \frac{2\pi}{\sqrt{u_0^2+v_0^2}},\; w_0 = \sqrt{u_0^2 + v_0^2}$
    - shifting of strides by distance d wrt origin & $\perp$ to ridges
    - $d = \frac{P\varphi}{2\pi} = \frac{\varphi}{\omega}$

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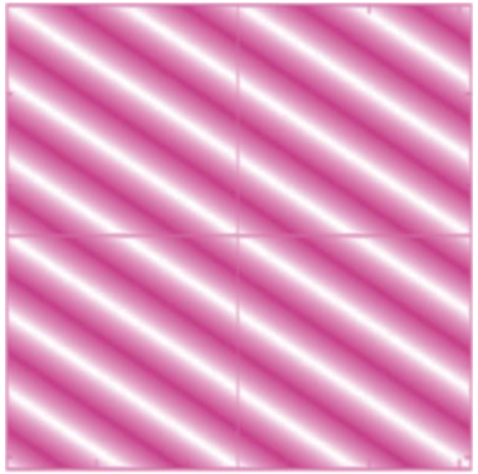

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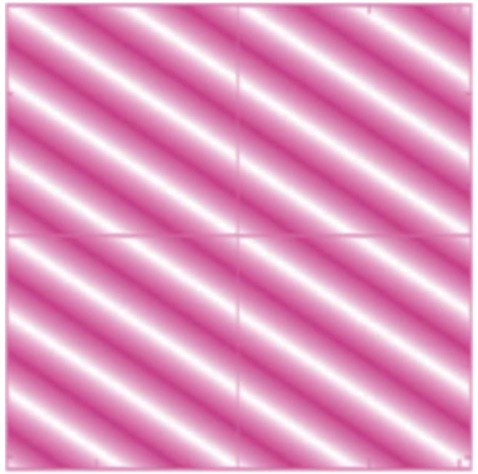

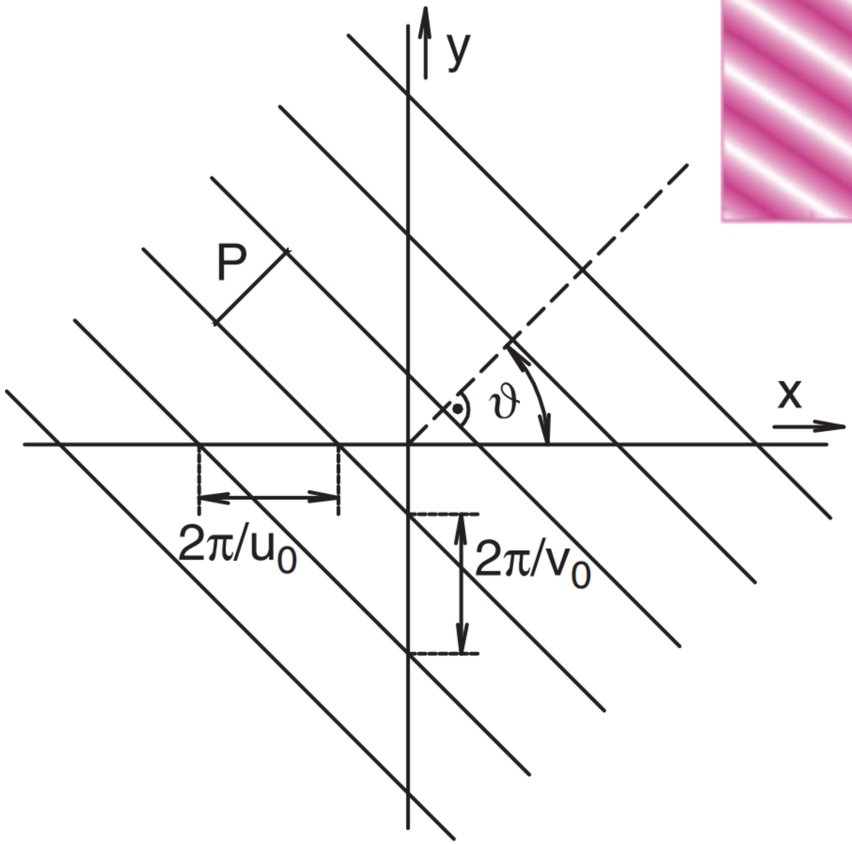

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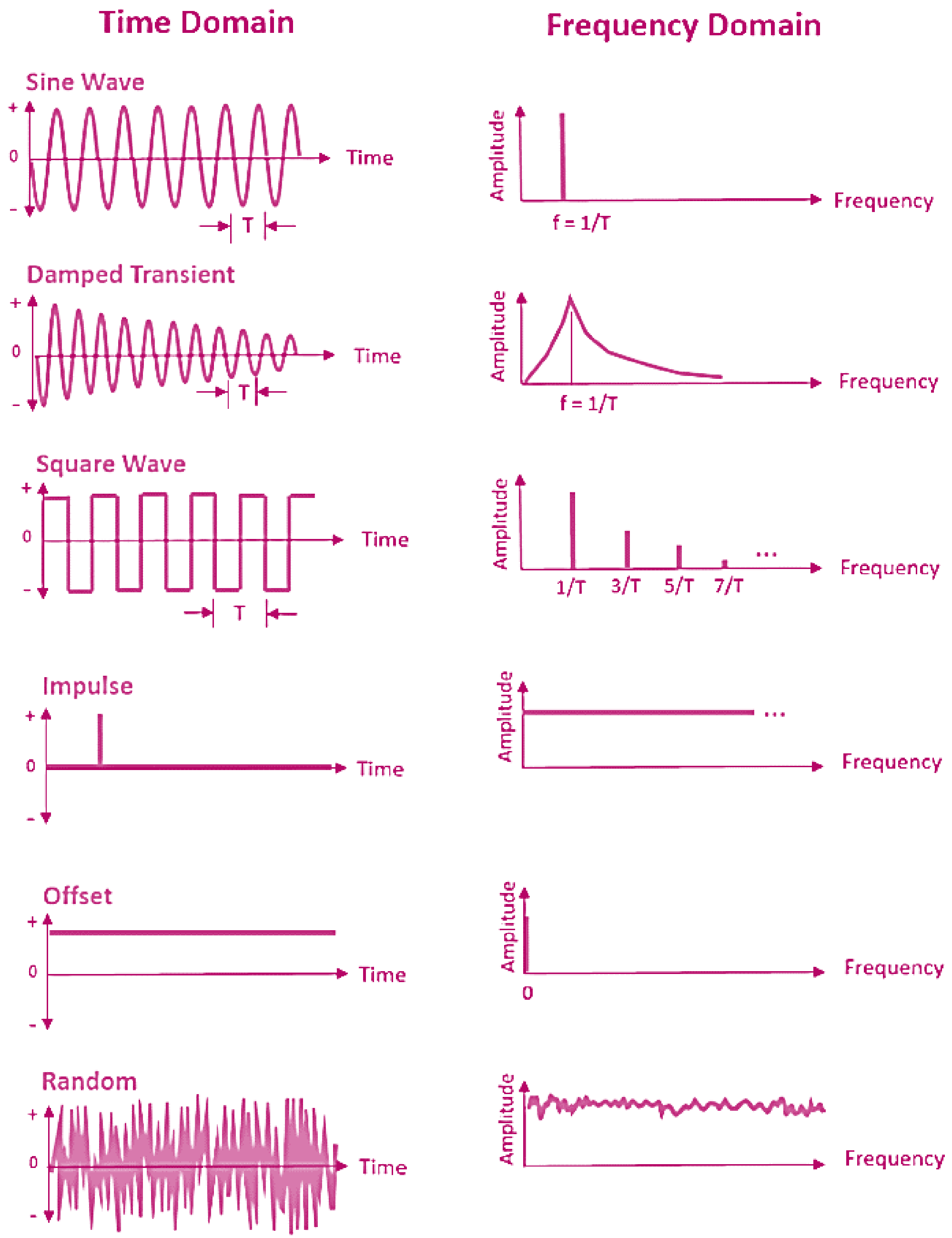
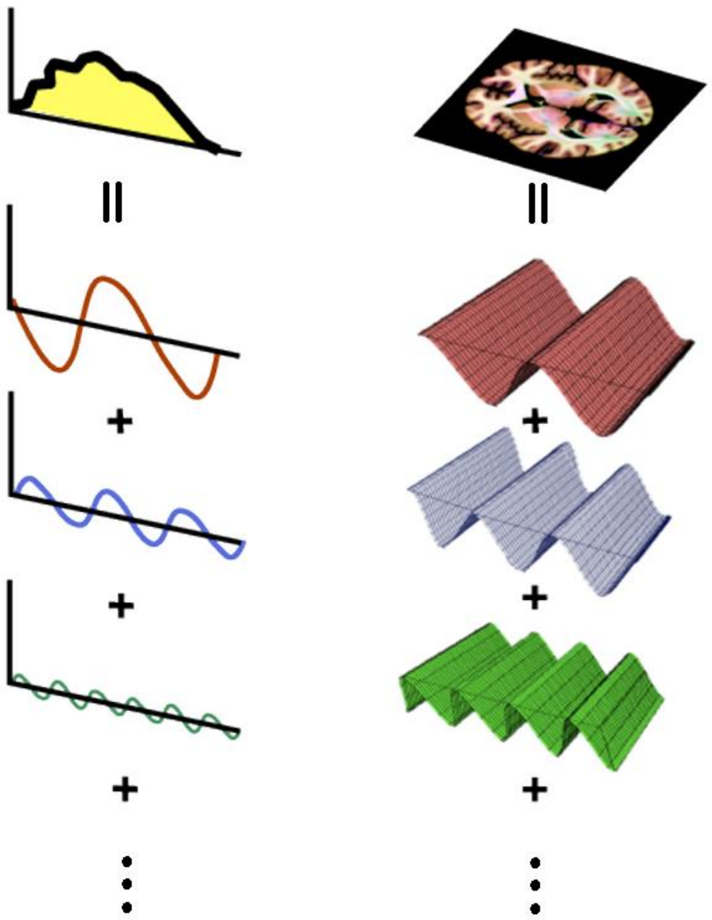
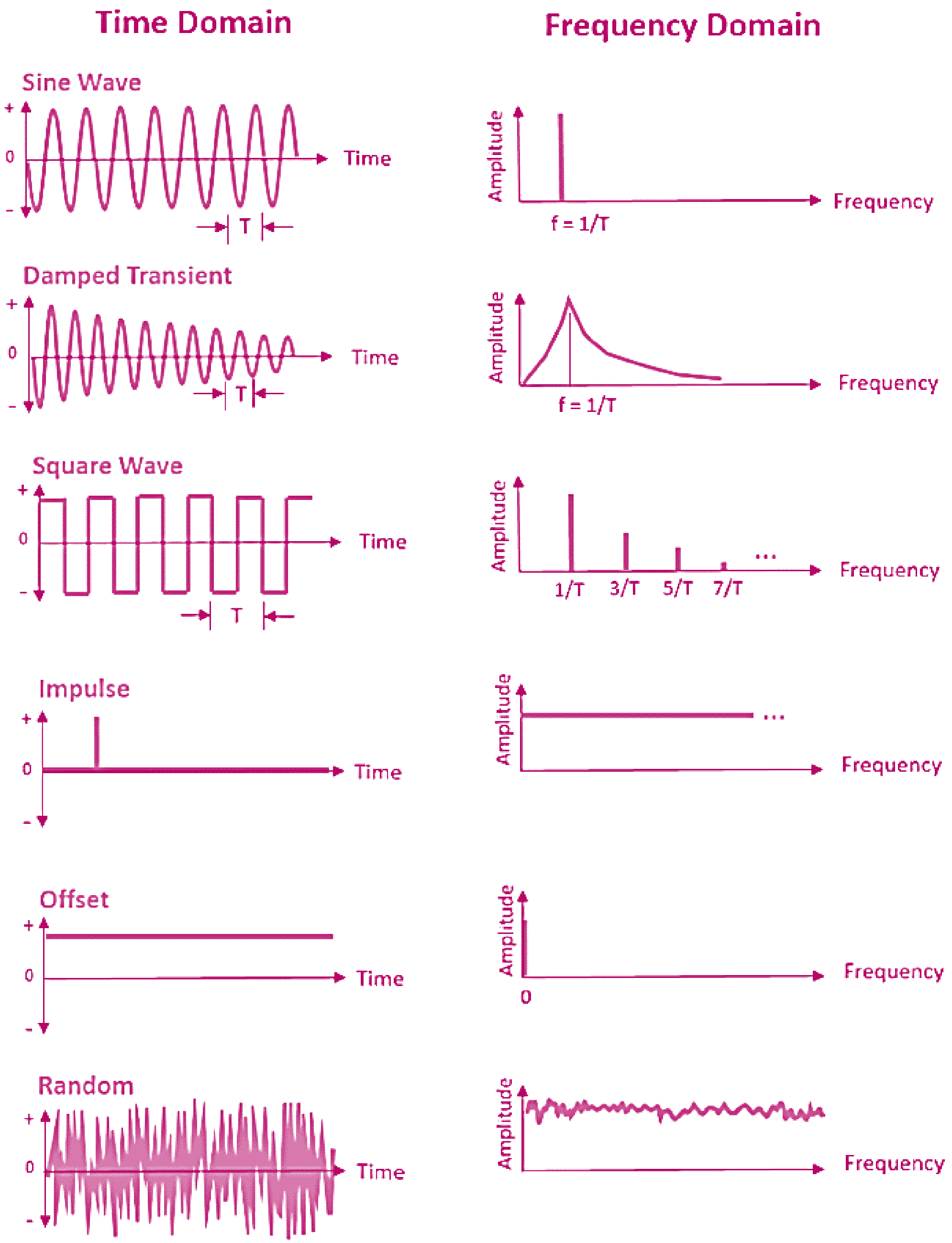

Time Domain
Frequency Domain
Sine Wave
+
0
-
Time
T
Amplitude
Frequency
f = 1/T
Damped Transient
+
0
-
Time
T
Amplitude
Frequency
f = 1/T
Square Wave
+
0
-
Time
T
Amplitude
Frequency
1/T
3/T
5/T
7/T
...
Impulse
+
0
-
Time
Amplitude
...
Frequency
Offset
+
0
-
Time
Amplitude
Frequency
0
Random
+
0
-
Time
Amplitude
Frequency

Frequency

- 1D
- 2D





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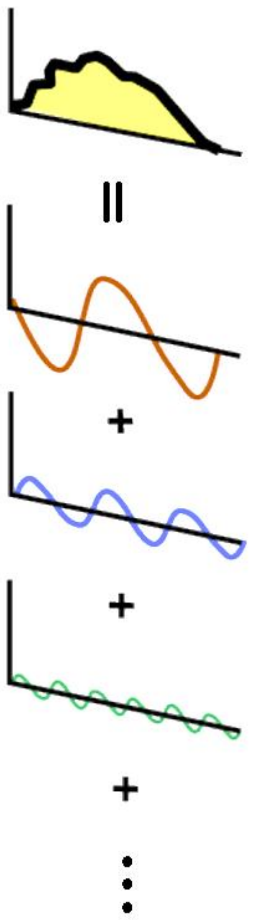

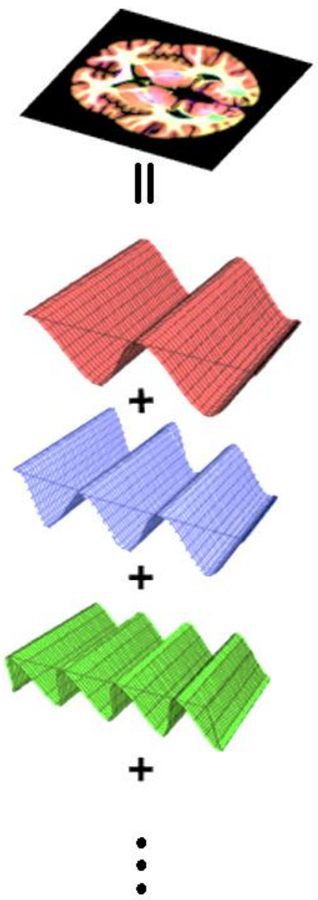

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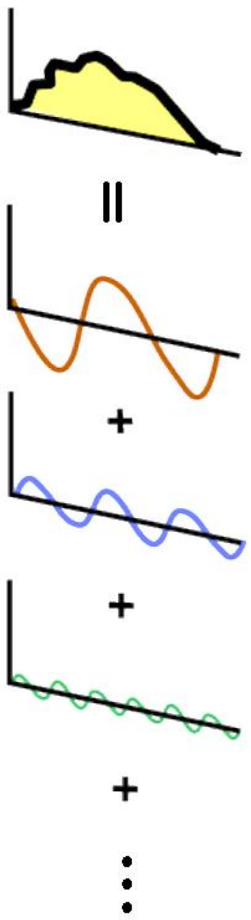

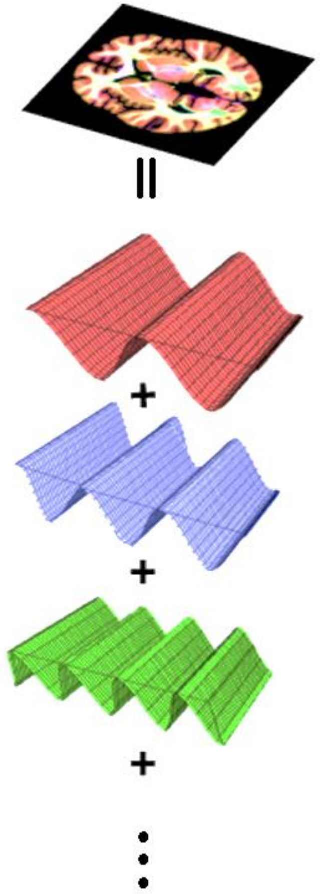

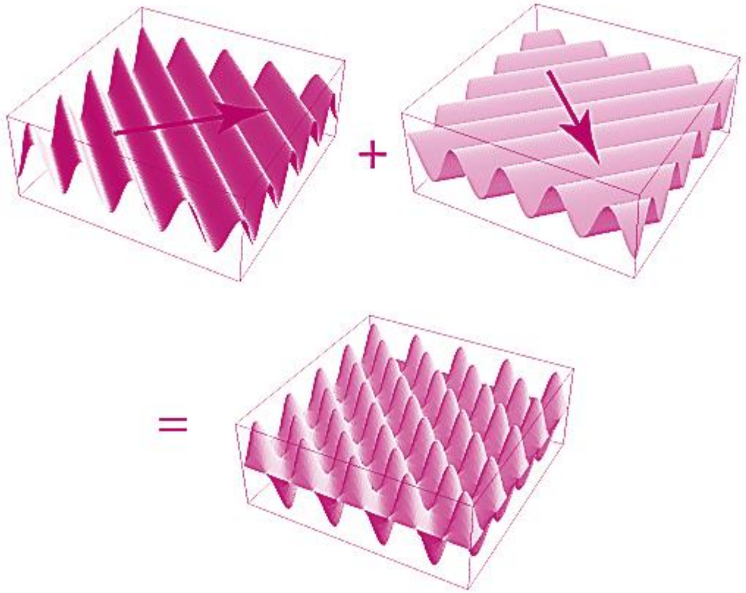

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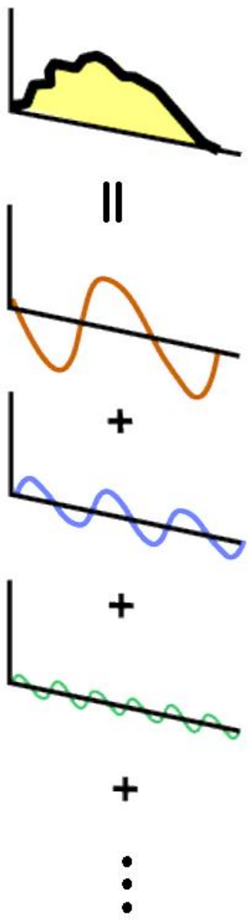

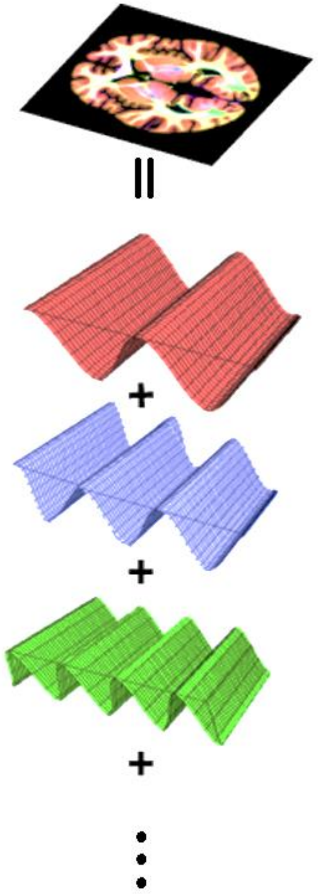

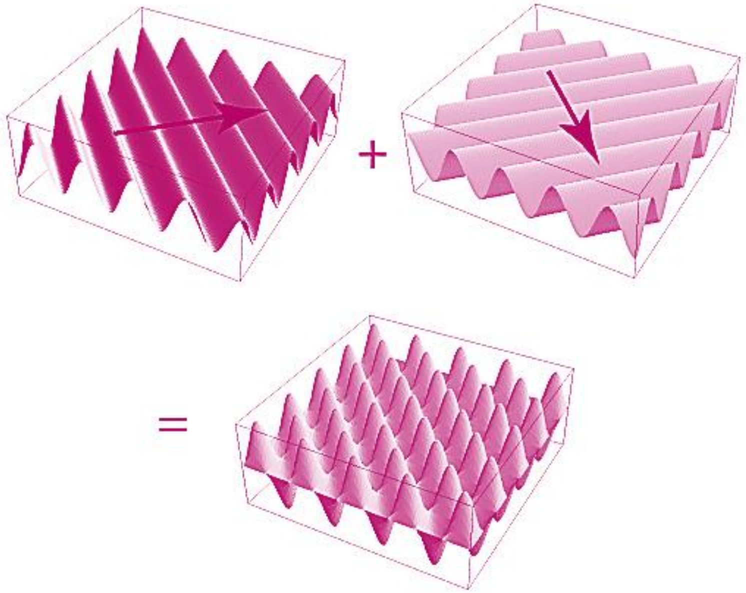

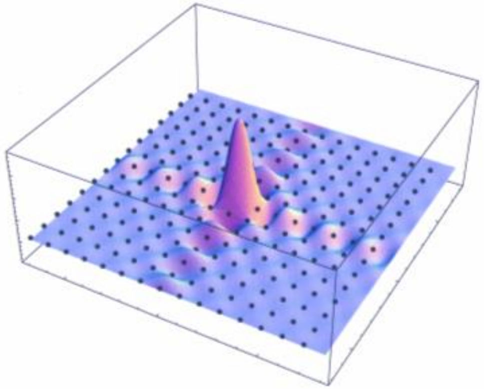

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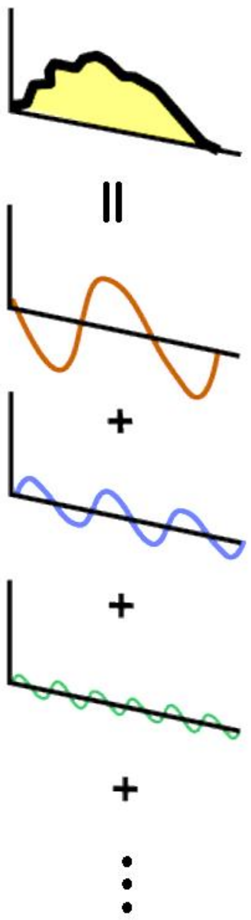
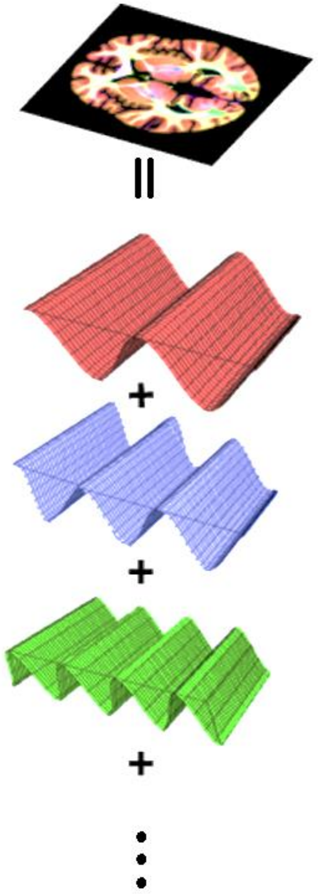
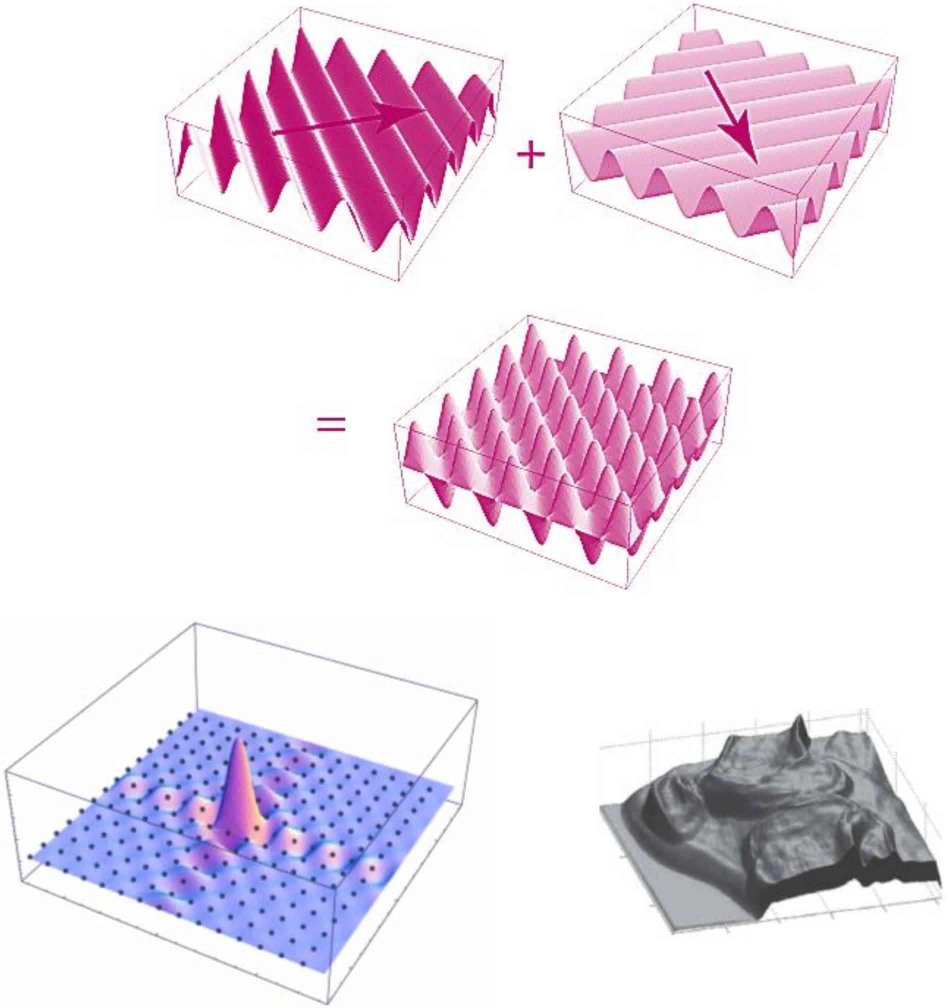


- 2D





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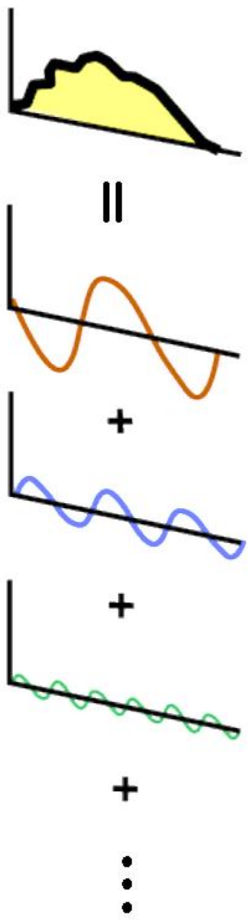

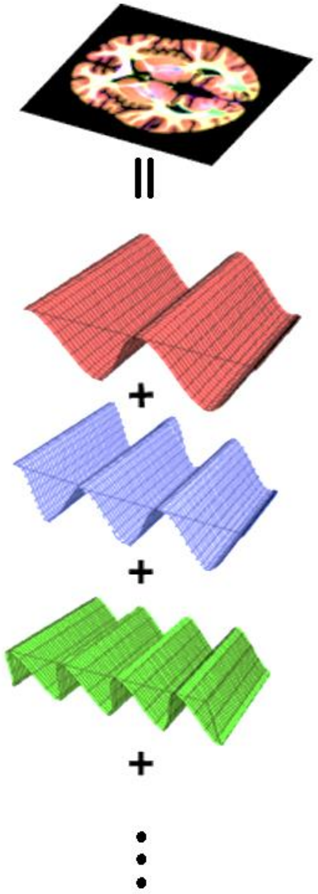

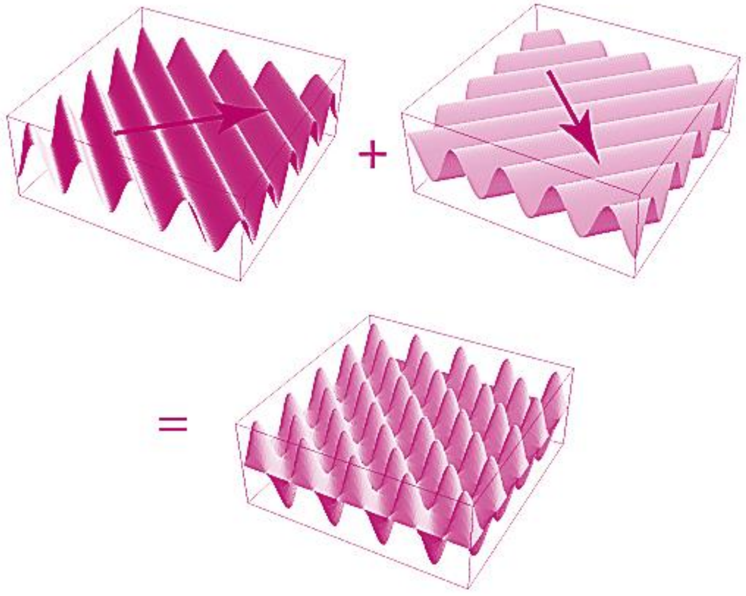

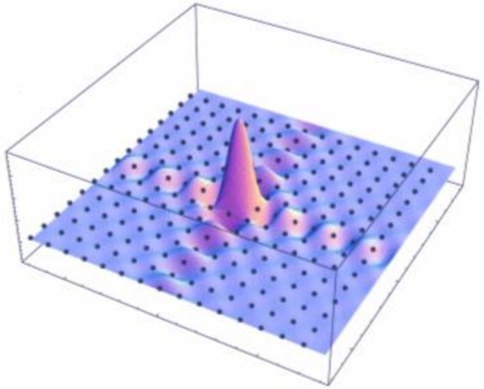

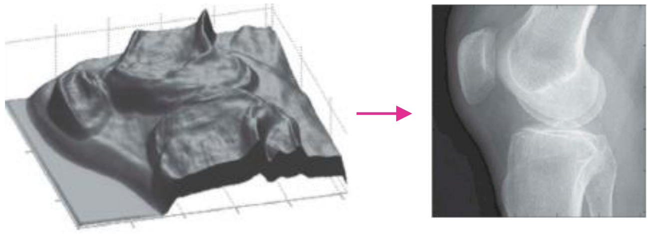

Fourier

- Fourier series
  - any periodic function can be approximated with series of harmonics
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Fourier

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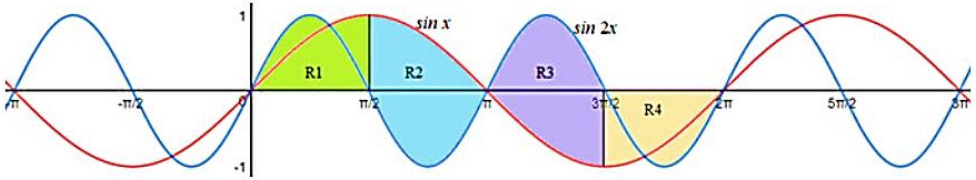

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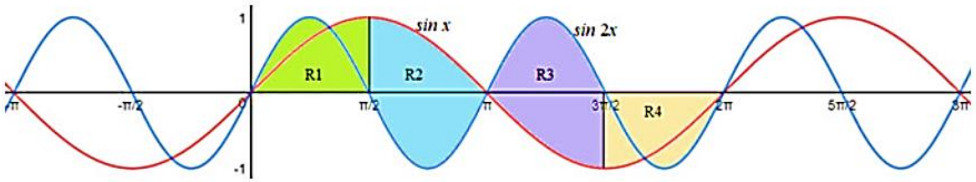

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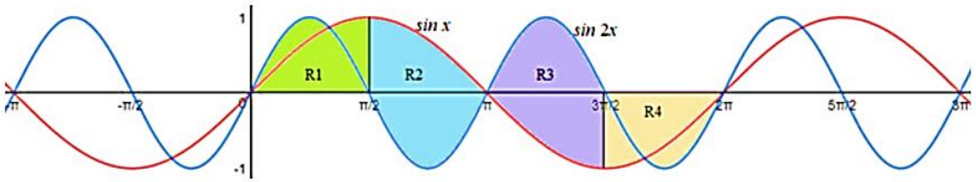

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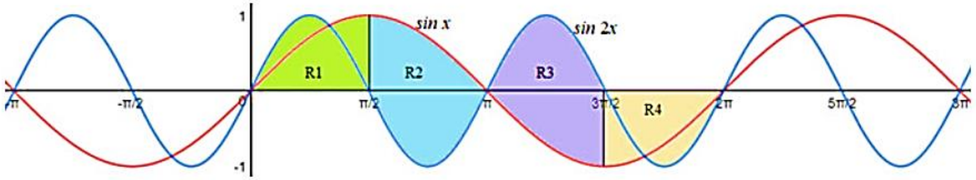

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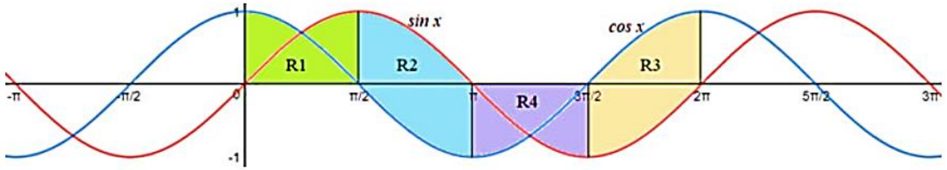

Fourier

- Fourier series
  - orthogonality

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Fourier

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  - equation

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Fourier

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  - Complex
  - $e^{i\theta} = ?$

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Fourier

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Fourier

- Fourier Transform
  - any signal
  - finite discontinuities in finite interval
  - $\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty$

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$L \to \infty$ Frequency n/L becomes continuous

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Fourier

- Discrete Fourier Transform
  - any digital signal
  - N samples in [0, L]
  - Δx sample step in x direction
  - $L = ?$

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-j\frac{2\pi nx}{L}}dx$$

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$$c_n = \frac{\Delta x}{N\Delta x}\sum_{k=0}^{N-1} f(k)e^{-j\frac{2\pi nk\Delta x}{N\Delta x}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f(k)e^{-j\frac{2\pi nk}{N}} n = 0, 1, 2, \ldots, N-1$$

Fourier

- Discrete Fourier Transform
  - DFT
  - N samples in [0, L]
  - $\frac{1}{N}$ outside scaling constant is interchangeably used either in IDFT or in DFT (like below)

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Fourier

- 2D DFT
  - MxN samples

$$0 \leq x < M$$
$$0 \leq y < N$$

  - image $f(x, y)$

Fourier

- 2D DFT
  - MxN samples
  - image $f(x, y)$

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

Fourier

- 2D DFT
  - MxN samples
  - image $f(x,y)$

❑ DFT: $$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

Fourier

- 2D DFT
  - MxN samples
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$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M + vy/N)}$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi(ux/M + vy/N)}$$

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Fourier

- 2D DFT
  - separability

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

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- 2D DFT
  - separability

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

$$F(u,v) = \sum_{x=0}^{M-1} \left[\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N} \right] e^{-j2\pi ux/M}$$

Fourier

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  - separability

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Fourier

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$$F(u,v) = FT_x \left\{ FT_y \left[f(x,y) \right] \right\}$$

Fourier

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  - separability

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$$F(u,v) = FT_x \left\{ FT_y \left[f(x,y) \right] \right\}$$

FT_x and FT_y are the 1D FTs on row and column, respectively.

Fourier rotation

- Image rotation

Fourier rotation

- Image rotation

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

Fourier rotation

- Image rotation

$$x' = x\cos(\theta) - y\sin(\theta)$$
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Inverse rotation $(-\theta)$

Fourier rotation

- Image rotation

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$$y' = x\sin(\theta) + y\cos(\theta)$$

Inverse rotation $(-\theta)$

$$x = +x'\cos(\theta) + y'\sin(\theta)$$
$$y = -x'\sin(\theta) + y'\cos(\theta)$$

Fourier rotation

- Image rotation

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Inverse rotation $(-\theta)$

$$x = +x'\cos(\theta) + y'\sin(\theta)$$
$$y = -x'\sin(\theta) + y'\cos(\theta)$$

$$g_r(x, y) =$$

$$f(x\cos(\theta) + y\sin(\theta), -x\sin(\theta) + y\cos(\theta))$$

Fourier rotation

- Image rotation

$$G_r(\Omega_1, \Omega_2) = \iint_{-\infty}^{\infty} g_r(x, y) e^{-j(\Omega_1 x + \Omega_2 y)} dx dy$$

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$$= \iint f(x\cos(\theta) + y\sin(\theta), -x\sin(\theta) + y\cos(\theta)) e^{-j(\Omega_1 x + \Omega_2 y)}$$

Inverse rotation $(-\theta)$

$$= \iint f(x', y') e^{-j[\Omega_1(x'\cos(\theta) - y'\sin(\theta)) + \Omega_2(x'\sin(\theta) + y'\cos(\theta))]} dx' dy'$$

$$x = +x'\cos(\theta) + y'\sin(\theta)$$
$$y = -x'\sin(\theta) + y'\cos(\theta)$$

$$= \iint f(x', y') e^{-j[(\Omega_1\cos(\theta) + \Omega_2\sin(\theta))x' + (-\Omega_1\sin(\theta) + \Omega_2\cos(\theta))y']} dx' dy'$$

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$$= \iint f(x\cos(\theta) + y\sin(\theta), -x\sin(\theta) + y\cos(\theta))e^{-j(\Omega_1 x + \Omega_2 y)}$$

$$= \iint f(x',y')e^{-j[\Omega_1(x'\cos(\theta) - y'\sin(\theta)) + \Omega_2(x'\sin(\theta) + y'\cos(\theta))]}dx'dy'$$

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$$= \iint f(x',y')e^{-j[\Omega_1(x'\cos(\theta)-y'\sin(\theta))+\Omega_2(x'\sin(\theta)+y'\cos(\theta))]}dx'dy'$$

$$= \iint f(x',y')e^{-j[(\Omega_1\cos(\theta)+\Omega_2\sin(\theta))x'+(-\Omega_1\sin(\theta)+\Omega_2\cos(\theta))y']}dx'dy'$$

$$= F(\Omega_1\cos(\theta) + \Omega_2\sin(\theta), -\Omega_1\sin(\theta) + \Omega_2\cos(\theta))$$

$$\boxed{G_r(\Omega_1,\Omega_2) = F(\Omega_1\cos(\theta) + \Omega_2\sin(\theta), -\Omega_1\sin(\theta) + \Omega_2\cos(\theta))}$$

Fourier

- 2D DFT
  - image
  - use grayscale image for DFT

Fourier

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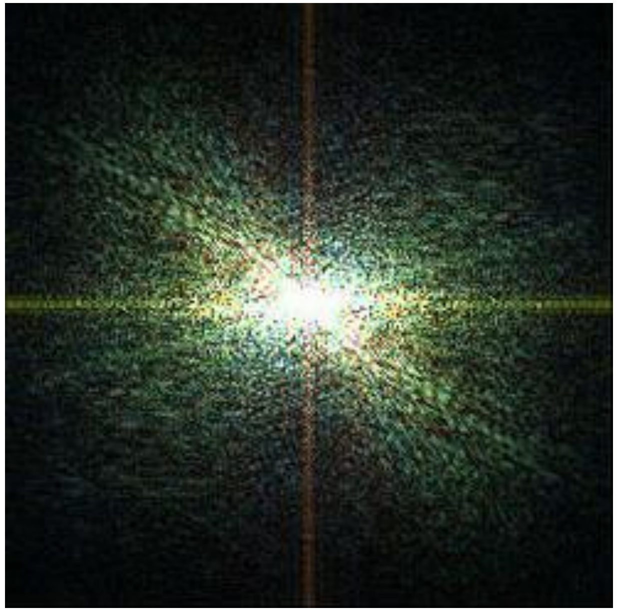
  - image
  - use grayscale image for DFT

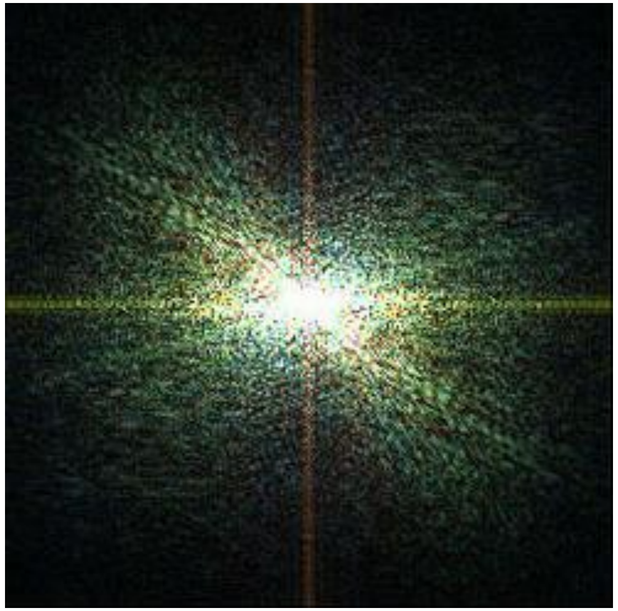
→



Fourier

- 2D DFT
  - image
  - use grayscale image for DFT

$f(x, y)$



Fourier

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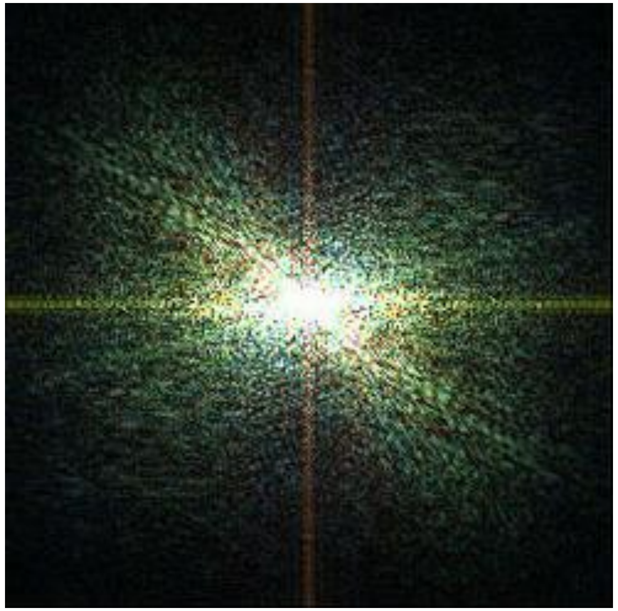
$f(x, y)$

$\longrightarrow$

Enhanced($|F(u, v)|$)



Fourier

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  - image

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  - image

$$f(x,y)$$

Fourier

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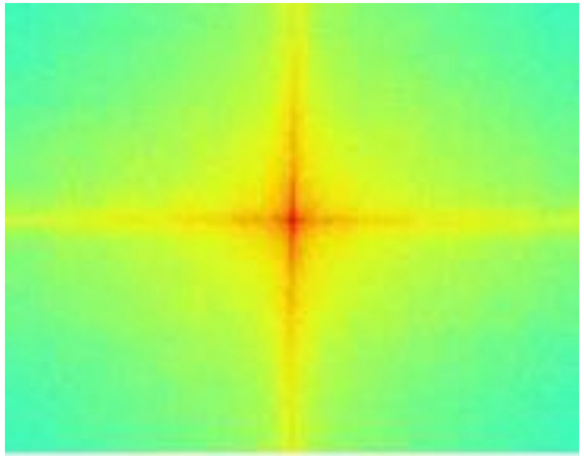
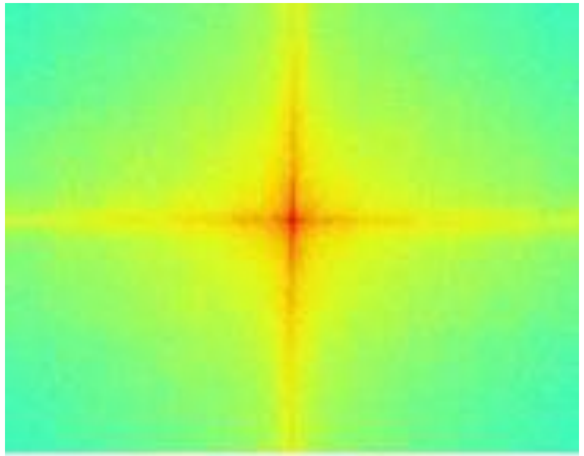


Fourier

- 2D DFT
  - image

$f(x, y)$

$|F(u, v)|$



Fourier

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  - image

$f(x,y)$

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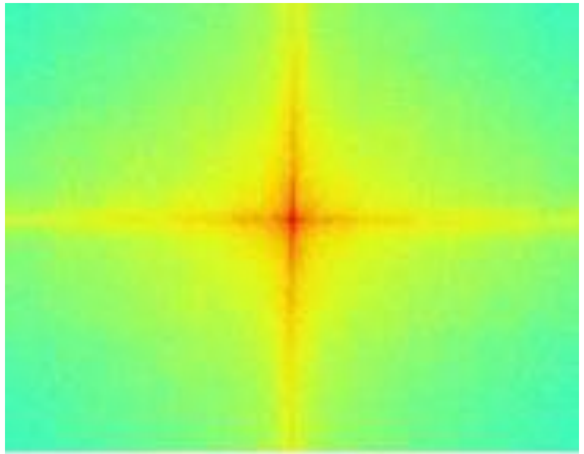
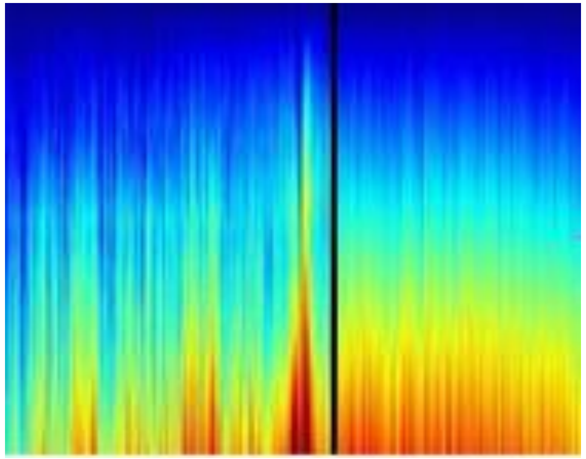
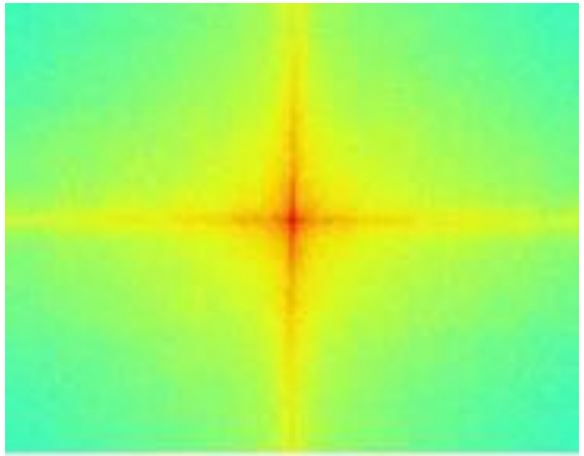


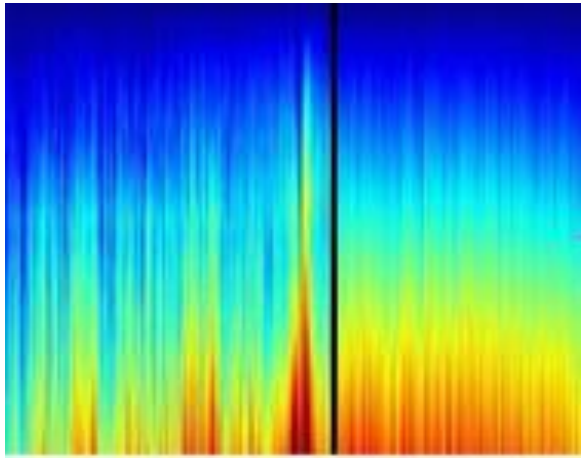


Fourier

- 2D DFT
  - image

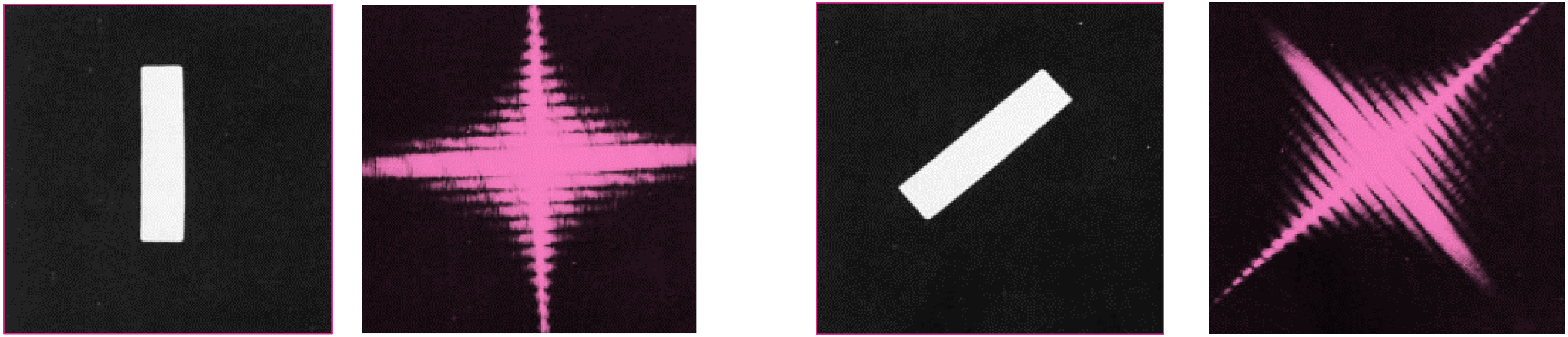
$f(x, y)$ $\qquad$ $|F(u, v)|$ $\qquad$ $\angle F(u, v)$





Fourier

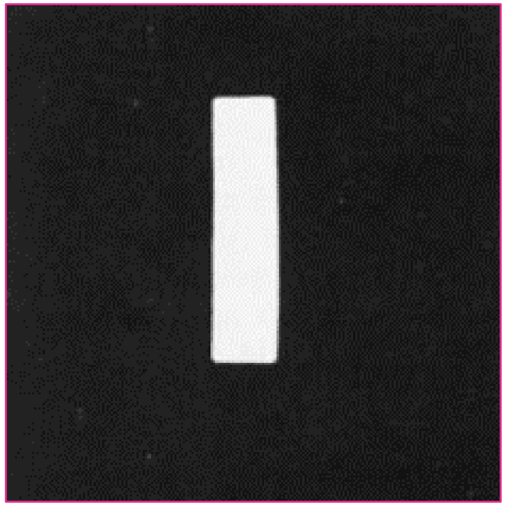
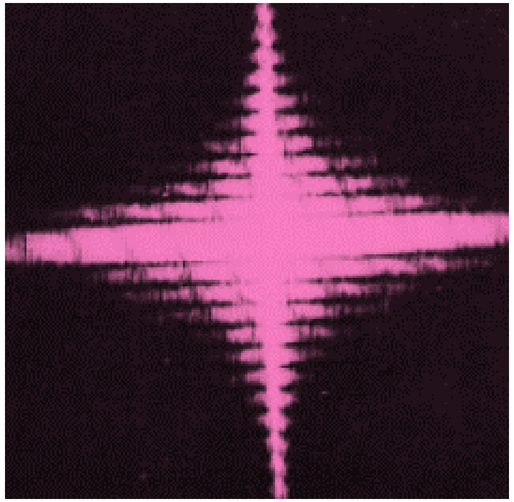
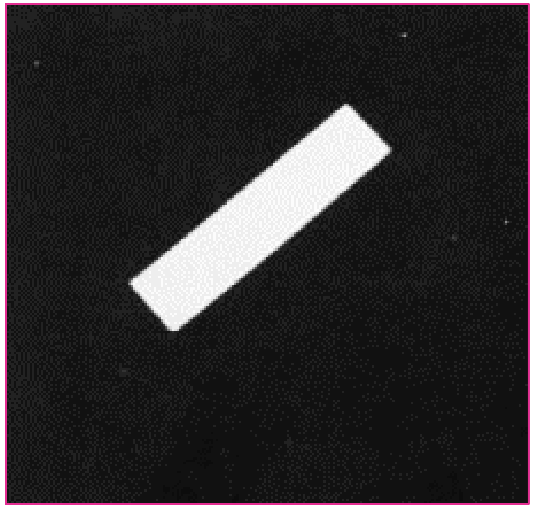
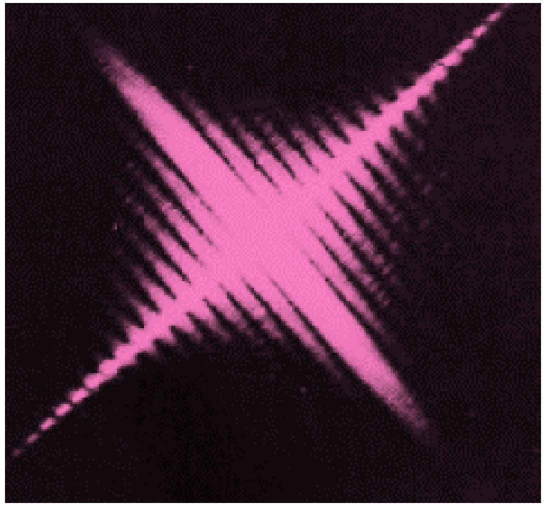
- 2D DFT
  - Image rotation



Fourier

- 2D DFT
  - Image rotation

$f(x, y)$

Fourier

- 2D DFT
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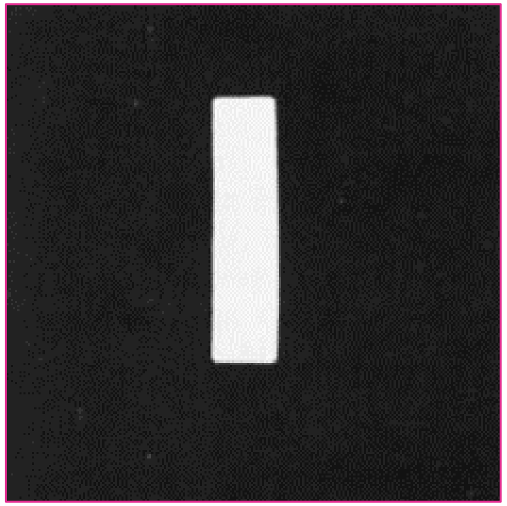
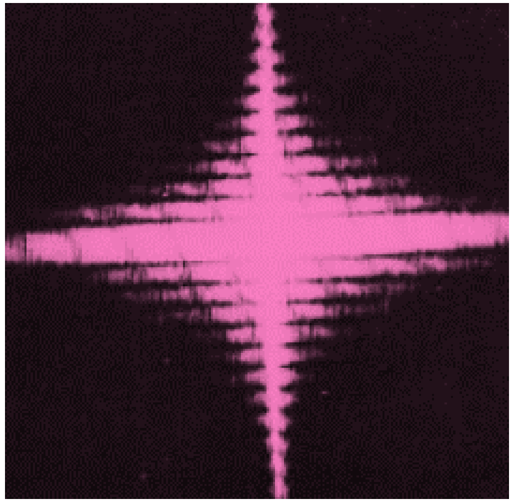
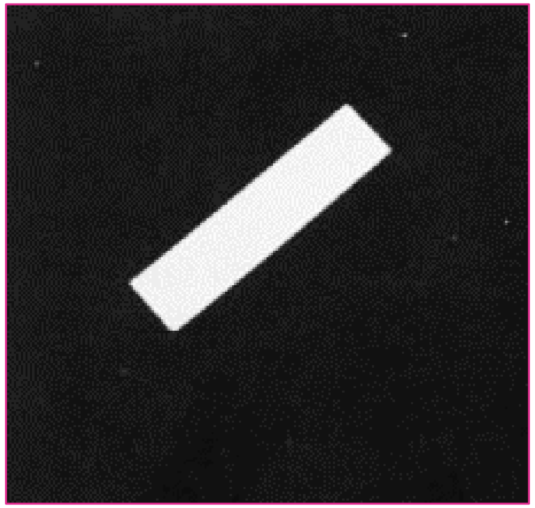
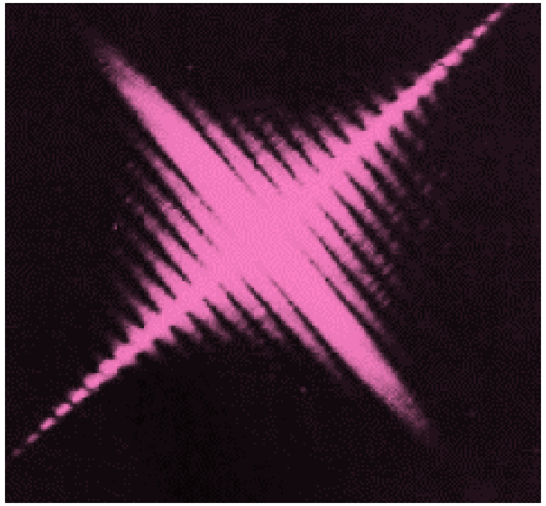
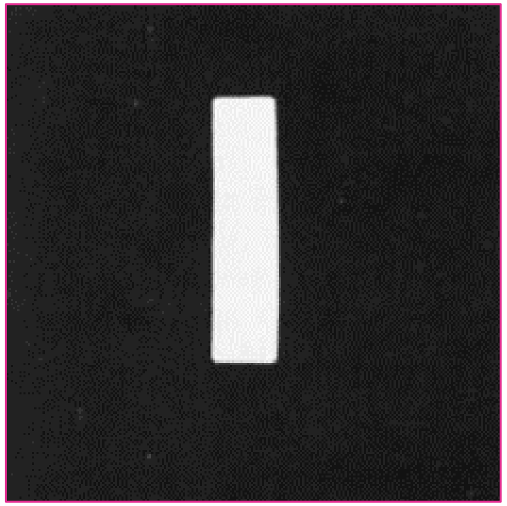
$f(x,y)$ $|F(u,v)|$

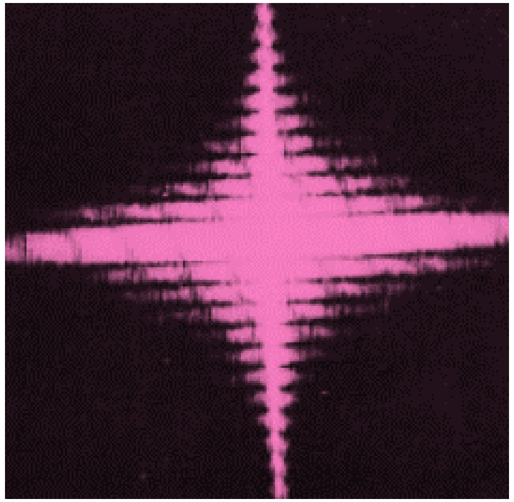
Fourier

- 2D DFT
  - Image rotation

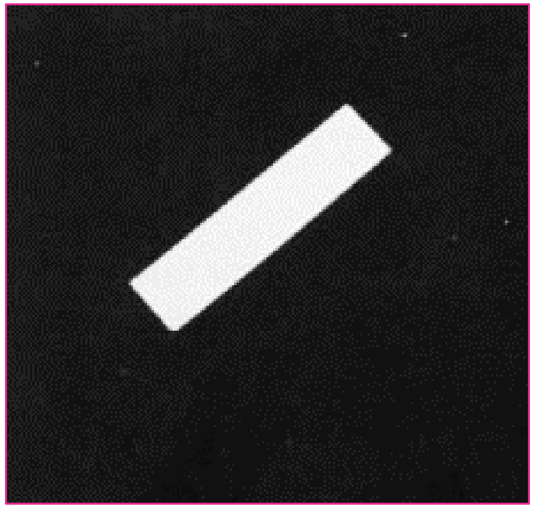
$f(x,y)$

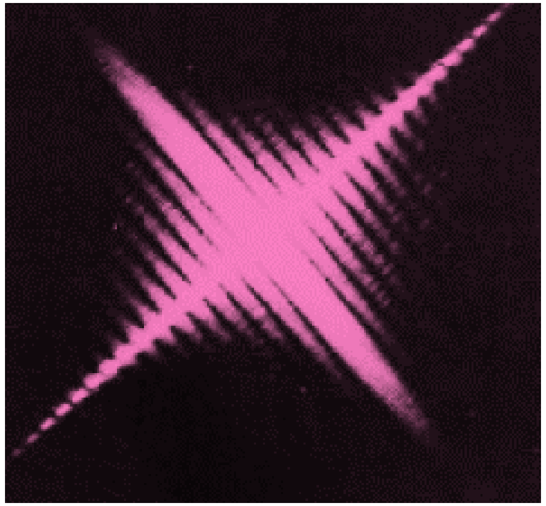


$|F(u,v)|$



$f(\angle x, \angle y)$





Fourier

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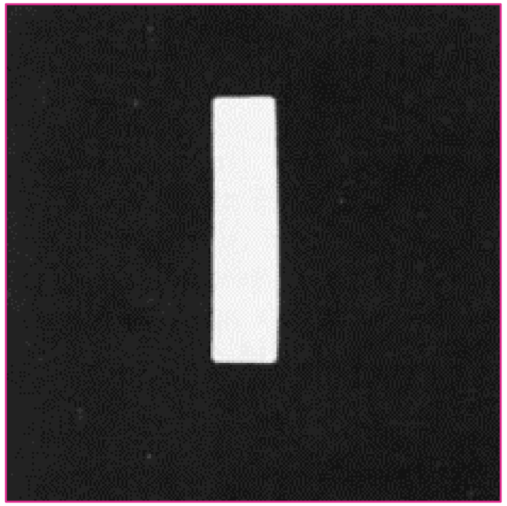
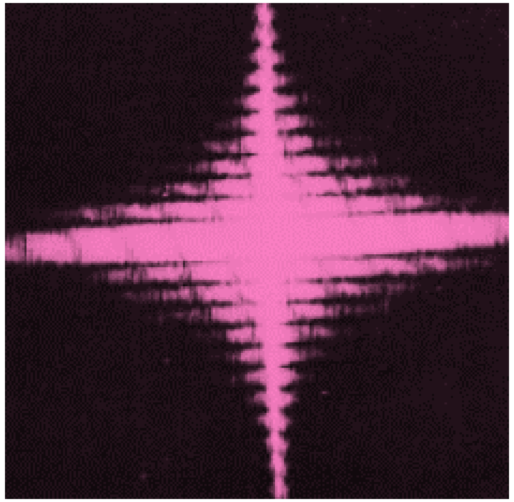


$|F(u, v)|$



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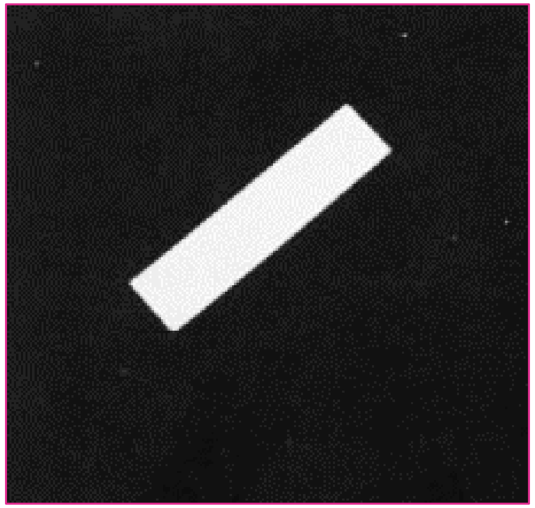
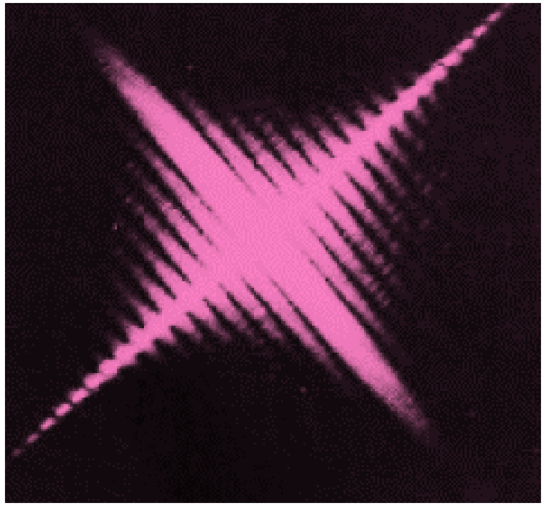


$|F(\angle u, \angle v)|$



Conclusion

- Frequency representation
- Fourier
  - Series
  - Transform
  - 2D DFT

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"Four years at IIT transform $\left(f \overset{*}{_F} T\right)$ life phenomenally"

-TS

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“Four years at IIT

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  - 2D DFT

"Four years at IIT transform $\left(f \, {}^{*}_{F} \, T\right)$ life phenomenally"

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Lagrange, Laplace, Monge